NOTES ON ABELIAN VARIETIES

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We fix a field k and an algebraic closure $\overline{k}$ of k. A variety over k is a geometrically integral and separated scheme of finite type over k. If X and Y are schemes over k, we denote by $X \times Y = X \times_{\operatorname{Spec} k} Y$, and Ω^1_X the sheaf of differential 1-forms on X relative to k.

1. Generalities on group schemes over a field

Definition 1.1. (i) A *group scheme over* k is a k-scheme $\pi : G \to \operatorname{Spec}(k)$ together with morphisms of k-schemes $m : G \times G \to G$ (multiplication), $i : G \to G$ (inverse), and $e : \operatorname{Spec}(k) \to G$ (identity section), such that the following conditions are satisfied:

$$m \circ (m \times \mathrm{Id}_G) = m \circ (\mathrm{Id}_G \times m) : G \times G \times G \to G,$$
$$m \circ (e \times \mathrm{Id}_G) = j_1 : \operatorname{Spec}(k) \times G \to G,$$
$$m \circ (\mathrm{Id}_G \times e) = j_2 : G \times \operatorname{Spec}(k) \to G,$$
$$e \circ \pi = m \circ (\mathrm{Id}_G \times i) \circ \Delta_G = m \circ (i \times \mathrm{Id}_G) \circ \Delta_G : G \to G,$$

where $j_1 : \operatorname{Spec}(k) \times G \xrightarrow{\sim} G$ and $j_2 : G \times \operatorname{Spec}(k) \xrightarrow{\sim} G$ are the natural isomorphisms.

(ii) A group scheme G over k is said to be *commutative* if, letting $s : G \times G \to G \times G$ be the isomorphism switching the two factors, we have the identity $m = m \circ s : G \times G \to G$.

(iii) A homomorphism of group schemes $f : G_1 \to G_2$ is a morphism of k-schemes which commutes with the morphisms of multiplication, inverse and identity section.

Remark 1.2. (i) For any k-scheme S, the set $G(S) = \mathrm{Mor}_{k-\mathrm{Sch}}(S, G)$ is naturally equipped with a group structure. By Yoneda Lemma, the group scheme G is completely determined by the functor $h_G \colon S \mapsto G(S)$ from the category of k-schemes to the category of groups. More precisely, the functor $G \mapsto h_G$ from the category of group schemes over k to the category $\mathrm{Funct}(k{-}\mathrm{Sch}, \mathrm{Group})$ of functors is fully faithful.

(ii) For any $n \in \mathbf{Z}$, we put $[n] = [n]_G : G \to G$ to be the morphism of k-schemes

$$G \xrightarrow{\Delta^{(n)}} \underbrace{G \times G \times \cdots \times G}_{n \text{ times}} \xrightarrow{m^{(n)}} G$$

if $n \geq 0$, and $[n] = [-n] \circ i$ if $n < 0$. If G is commutative, $[n]_G$ is a homomorphism of group schemes. Moreover, G is commutative if and only if i is a homomorphism.

Example 1.3. (1) The additive group. Let $\mathbf{G}_a = \operatorname{Spec}(k[X])$ be the group scheme given by

$$m^* : k[X] \to k[X] \otimes k[X] \quad X \mapsto X \otimes 1 + 1 \otimes X$$
$$i^* : k[X] \to k[X] \quad X \mapsto -X$$
$$[n]_{\mathbf{G}_a} : k[X] \to k[X] \quad X \mapsto nX.$$

For any k-scheme S, $\mathbf{G}_a(S) = \mathrm{Hom}_{k-\mathrm{Alg}}(k[X], \Gamma(S, \mathcal{O}_S)) = \Gamma(S, \mathcal{O}_S)$ with the additive group law.

(2) The multiplicative group is the group scheme $\mathbf{G}_m = \mathrm{Spec}(k[X, 1/X])$ given by

$$m^*(X) = X \otimes X, \quad e^*(X) = 1, \quad i^*(X) = 1/X.$$

For any k-scheme S, we have $\mathbf{G}_m(S) = \Gamma(S, \mathcal{O}_S)^\times$ with the multiplicative group law.

(3) For any integer $n > 0$, the closed subscheme $\mu_n = \mathrm{Spec}(k[X]/(X^n - 1))$ of $\mathbf{G}_m$ has a group structure induced by that of $\mathbf{G}_m$. For any k-scheme S, $\mu_n(S)$ is the group of n-th roots of unity in $\Gamma(S, \mathcal{O}_S)^\times$, *i.e.*,

$$\mu_n(S) = \{f \in \Gamma(S, \mathcal{O}_S)^\times \mid f^n = 1\}.$$

We note that μ_n is not reduced if the characteristic of k divides n.

(4) For $n \in \mathbf{Z}_{\geq 1}$, we put $\mathrm{GL}_n = \mathrm{Spec}(k[(T_{i,j})_{1 \leq i,j \leq n}, U]/(U \det(T_{i,j}) - 1)$. It is endowed with a group scheme structure by imposing

$$m^*(T_{i,j}) = \sum_{k=1}^{n} T_{i,k} \otimes T_{k,j} \quad e^*(T_{i,j}) = \delta_{i,j},$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. An explicit formula for the coinverse i^* is more complicated, and it can be given by the Cramer's rule for the inverse of a square matrix. For each S, $\mathrm{GL}_n(S)$ is the general linear group with coefficients in $\Gamma(S, \mathcal{O}_S)$. We have of course $\mathrm{GL}_1 = \mathbf{G}_m$.

Proposition 1.4. *Any group scheme over k is separated.*

Proof. This follows from the Cartesian diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad \pi \quad} & \mathrm{Spec}(k) \\ \downarrow{\scriptstyle \Delta_G} & & \downarrow{\scriptstyle e} \\ G \times G & \xrightarrow{m \circ (\mathrm{Id}_G \times i)} & G \end{array}$$

and the fact that e is a closed immersion. □

Lemma 1.5. *Let X be a geometrically connected (resp. geometrically irreducible, resp. geometrically reduced) k-scheme, Y be a connected (resp. irreducible, resp. reduced) k-scheme. Then $X \times Y$ is connected (resp. irreducible, resp. reduced).*

For a proof, see [EGA IV, 4, 5].

Proposition 1.6. *Let G be a group scheme over k. If k is perfect, then the reduced subscheme $G_{\mathrm{red}} \subset G$ is a closed subgroup scheme of G.*

Proof. Since k is perfect, the product $G_{\mathrm{red}} \times G_{\mathrm{red}}$ is still reduced by 1.5. The composed morphism $G_{\mathrm{red}} \times G_{\mathrm{red}} \hookrightarrow G \times G \xrightarrow{m} G$ factors through G_{red}. This gives G_{red} a subgroup scheme structure of G. □

Remark 1.7. If k is imperfect, the analogue of 1.6 is not true in general. In fact, if a is an element of k which is not a p-th power, where $p = \mathrm{char}(k)$, then $G = \mathrm{Spec}(k[X]/(X^{p^2} - aX^p))$ is a subgroup scheme of $\mathbf{G}_a$, but $G_{\mathrm{red}} = \mathrm{Spec}(k[X]/(X(X^{p^2 - p} - a)))$ is not a subgroup scheme of G.

Lemma 1.8. *Let X be a connected scheme over k with a rational point $x \in X(k)$. Then X is geometrically connected.*

Proof. This is [EGA IV 4.5.14]. □

Proposition 1.9. *Let G be a group scheme, locally of finite type over k, and G^0 be the connected component of G containing $e \in G(k)$.*

(i) The following properties are equivalent:

(a1) $G \otimes_k K$ is reduced for some perfect field extension K/k;

(a2) the ring $\mathcal{O}_{G,e} \otimes_k K$ is reduced for some perfect field extension K/k;

(b1) G is smooth over k;

(b2) G is smooth over k at e;

(ii) The identity component G^0 is actually an open and closed subgroup scheme of G, geometrically irreducible. In particular, we have $(G_K)^0 = (G^0)_K$ for any field extension K/k.

(iii) Every connected component of G is irreducible and of finite type over k.

Remark 1.10. (i) A reduced group scheme over k is not necessarily smooth unless k is perfect. In fact, let k be an imperfect field of characteristic p, α be an element of k which is not a p-th power. Consider the subgroup scheme $G = \mathrm{Spec}(k[X,Y]/(X^p + \alpha Y^p))$ of $\mathrm{Spec}(k[X,Y]) \simeq \mathbf{G}_a \times \mathbf{G}_a$. Then G is regular but not smooth over k. In fact, $G \otimes_k k(\sqrt[p]{\alpha})$ is not reduced.

(ii) The non-neutral components of a group scheme over k are not necessarily geometrically irreducible. Consider for example a prime number p invertible in k. Then the number of irreducible components of μ_p is 2 if k does not contain any p-th root of unity different from 1, and is p otherwise. In particular, $\mu_{p,\mathbf{Q}}$ has exactly 2 irreducible components while $\mu_{p,\mathbf{Q}(\zeta_p)}$ has exactly p irreducible components, where ζ_p is a primitive p-th root of unity.

Proof. (i) We only need to prove the implication $(a2) \Rightarrow (b1)$. We may assume $k = \overline{k}$. For $g \in G(k)$, we denote by $r_g : G \to G$ the right translation by g. It's clear that r_g induces an isomorphism of local rings $\mathcal{O}_{G,g} \simeq \mathcal{O}_{G,e}$. Hence $(a2)$ implies that G is reduced. Let $\mathrm{sm}(G) \subset G$ be the smooth locus. This is a Zariski dense open subset of G, stable under all the translations r_g. Hence we have $\mathrm{sm}(G) = G$.

(ii) By Lemma 1.8, G^0 is geometrically connected. Hence so is $G^0 \times G^0$ by 1.5. So under the multiplication morphism of G, the image of $G^0 \times G^0$ lies necessarily in G^0. This shows that G^0 is a closed subgroup scheme of G.

Next we show that G^0 is geometrically irreducible and quasi-compact. Since G^0 is stable under base field extensions, we may assume $k = \overline{k}$. Since G^0 is irreducible if and only if G^0_{red} is, we may assume that G^0 is reduced. By (ii), this implies that G^0 is smooth. It's well known that a smooth variety is connected if and only if it's irreducible. To prove the quasi-compactness of G^0, we take a non-empty affine open subset $U \subset G^0$. Then U is dense in G^0, since G^0 is irreducible. For every $g \in G^0(k)$, the two open dense subsets gU^{-1} and U have non-trivial intersection. Hence the map $U \times U \to G^0$ given by multiplication is surjective. Since $U \times U$ is quasi-compact, so is G^0.

(iii) Again we may assume $k = \overline{k}$. Then every connected component of G is the right translation of G^0 by a rational point. □

Let G be a group scheme, locally of finite type over k, and $\hat{G}$ be the completion of G along the identity section e. The group law of G induces a (formal) group law on $\hat{G}$, *i.e.,* we have a co-multiplication map

$$\hat{m}^* : \hat{\mathcal{O}}_{G,e} \to \hat{\mathcal{O}}_{G,e} \hat{\otimes} \hat{\mathcal{O}}_{G,e} \tag{1.10.1}$$

where $\hat{\mathcal{O}}_{G,e}$ is the completion of $\mathcal{O}_{G,e}$. In particular, for any $n \in \mathbf{Z}_{\geq 1}$, we have a natural map $\hat{m}^* : \mathcal{O}_{G,e} \to (\mathcal{O}_{G,e}/\mathfrak{m}^n) \otimes (\mathcal{O}_{G,e}/\mathfrak{m}^n)$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{G,e}$.

Theorem 1.11 (Cartier)**.** *Let G be a group scheme, locally of finite type over k. If k has characteristic 0, then G is reduced, hence smooth over k.*

Proof. (Oort) Let $A = \mathcal{O}_{G,e}$, $\mathfrak{m} \subset A$ be the maximal ideal, and $\mathrm{nil}(A) \subset A$ be the nilradical. Since k is perfect, G_{red} is a closed subgroup scheme of G. It follows thus from Proposition 1.9(ii) that $A_{\mathrm{red}} = A/\mathrm{nil}(A)$ is a regular local ring. Let $\mathfrak{m}_{\mathrm{red}} \subset A_{\mathrm{red}}$ be the maximal ideal of A_{red}. Then we have

$$\dim(A) = \dim(A_{\mathrm{red}}) = \dim_k(\mathfrak{m}_{\mathrm{red}}/\mathfrak{m}_{\mathrm{red}}^2) = \dim_k\big(\mathfrak{m}/(\mathfrak{m}^2 + \mathrm{nil}(A))\big).$$

Thus it suffices to show that $\mathrm{nil}(A) \subset \mathfrak{m}^2$. Since then, we will have $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = \dim(A)$, hence A is a regular local ring.

Let $0 \neq x \in \mathrm{nil}(A)$, and $n \geq 2$ be the positive integer such that $x^{n-1} \neq 0$ and $x^n = 0$. Since A is noetherian, there exists an integer $q \geq 2$ with $x^{n-1} \notin \mathfrak{m}^q$. We put $B = A/\mathfrak{m}^q$, $\mathfrak{m}_B = \mathfrak{m}/\mathfrak{m}^q$, and let $\overline{x}$ denote the image of x in B. As remarked above, the multiplication law of G induces a homomorphism $\hat{m}^* : A \to B \otimes B$. Since $e \in G(k)$ is a two-sided identity element, we have

$$\hat{m}^*(x) = \overline{x} \otimes 1 + 1 \otimes \overline{x} + y \quad \text{with } y \in \mathfrak{m}_B \otimes \mathfrak{m}_B.$$

From $x^n = 0$, we get

$$0 = \hat{m}^*(x^n) = \hat{m}^*(x)^n = (\overline{x} \otimes 1 + 1 \otimes \overline{x} + y)^n,$$

hence

$$n \cdot (\overline{x}^{n-1} \otimes \overline{x}) \in \big((\overline{x}^{n-1}\mathfrak{m}_B) \otimes \mathfrak{m}_B + \mathfrak{m}_B \otimes \mathfrak{m}_B^2\big).$$

Since $\mathrm{char}(k) = 0$, we have $(\overline{x}^{n-1} \otimes \overline{x}) \in (\overline{x}^{n-1}\mathfrak{m}_B) \otimes \mathfrak{m}_B + \mathfrak{m}_B \otimes \mathfrak{m}_B^2$. This implies that either $\overline{x}^{n-1} \in \overline{x}^{n-1}\mathfrak{m}_B$, or $\overline{x} \in \mathfrak{m}_B^2$. If it's the first case, Nakayama's lemma would imply that $\overline{x}^{n-1} = 0$. Hence we have $x \in \mathfrak{m}^2$.

□

Definition 1.12. Let G be a group scheme over k, and Ω^1_G be the sheaf of differential 1-forms on G with respect to k. A section $\alpha \in \Gamma(G, \Omega^1_G)$ is said to be *right invariant* (*resp. left invariant*), if we have $pr_1^*(\alpha) = m^*(\alpha)$ in $\Gamma(G \times G, pr_1^*\Omega^1_G)$ (*resp.* $pr_2^*(\alpha) = m^*(\alpha)$ in $\Gamma(G \times G, pr_2^*\Omega^1_G)$).

Remark 1.13. Let α be a right invariant differential 1-form of G. For each $g \in G(k)$, we denote by $r_g : G \to G$ the morphism of right translation by g. Since $pr_1 \circ (\mathrm{Id}_G \times (g \circ \pi)) = \mathrm{Id}_G$ and $m \circ (\mathrm{Id}_G \times (g \circ \pi)) = r_g$, we have $r_g^*(\alpha) = (\mathrm{Id}_G \times (g \circ \pi))^* m^* \alpha = (\mathrm{Id}_G \times (g \circ \pi))^* pr_1^* \alpha = \alpha$. Conversely, if $k = \overline{k}$ and $\alpha \in \Gamma(G, \Omega^1_G)$ is invariant under any r_g^*, then α is right invariant in sense of 1.12. We have similar remarks for left invariant 1-forms.

Proposition 1.14. *Let $\omega_G = e^*\Omega^1_G$ be the cotangent space of G at e. Then there is a canonical isomorphism $\pi^*\omega_G \simeq \Omega^1_G$ such that the induced adjunction map $\omega_G \to \Gamma(G, \Omega^1_G)$ is injective and identifies ω_G with the space of right invariant* 1*-forms of G.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} G & \xrightarrow{(e\circ\pi,\mathrm{Id}_G)} & G\times G & \xrightarrow{\tau} & G\times G & \xrightarrow{pr_2} & G \\ & & & \searrow^{m} & \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle \pi} \\ & & & & G & \xrightarrow{\pi} & \mathrm{Spec}(k), \end{array}$$

where τ is the isomorphism $(x, y) \mapsto (xy, y)$. If we consider $G \times G$ as a scheme over G via pr_2, then τ is a G-automorphism of $G \times G$. It induces an isomorphism of differential modules

$$\Omega^1_{G\times G/G} \simeq \tau^*\Omega^1_{G\times G/G}.$$

By base change formula for differential modules, we have $\Omega^1_{G\times G/G} \simeq pr_1^*\Omega^1_G$. Thus the above isomorphism gives rise to an isomorphism

$$pr_1^*\Omega^1_G \simeq \tau^* pr_1^*\Omega^1_G = m^*\Omega^1_G.$$

Pulling back by $(e\circ\pi, \mathrm{Id}_G)$, we get

$$\pi^*\omega_G = (e\circ\pi, \mathrm{Id}_G)^* pr_1^*\Omega^1_G \simeq (e\circ\pi, \mathrm{Id}_G)^* m^*\Omega^1_G = \Omega^1_G.$$

□

Corollary 1.15. *Let $f : \mathbf{P}^1_k \to G$ be a morphism from the projective line to a group scheme G over k. Then there exists a k-rational point $x \in G(k)$, such that $f(\mathbf{P}^1_k) = \{x\}$.*

Proof. It's clear that the image of $\mathbf{P}^1_k$ is either a curve or a k-rational point of G. If it were the first case, let X denote the image of $\mathbf{P}^1_k$, and $k(\mathbf{P}^1)$ and $k(X)$ be respectively the fraction fields of $\mathbf{P}^1_k$ and X. Then $k(\mathbf{P}^1_k)$ is a finite extension of $k(X)$. Assume first that the extension $k(\mathbf{P}^1_k)/k(X)$ is separable (this is automatic if $\mathrm{char}(k) = 0$). Then the morphism $f : \mathbf{P}^1_k \to X \subset G$ is generically étale, hence there exists a closed point $t \in \mathbf{P}^1_k$ such that the induced map $f^*\Omega^1_G \otimes \kappa(t) \to \Omega^1_{\mathbf{P}^1_k} \otimes \kappa(t)$ is surjective. But according to the previous proposition, Ω^1_G is generated by its global sections, so there exists a global section of $\Omega^1_{\mathbf{P}^1}$ that is non-vanishing at t. But this is absurd, since $\Omega^1_{\mathbf{P}^1_k} \simeq \mathcal{O}_{\mathbf{P}^1_k}(-2)$ does not have any non-zero global sections at all! In the general case, we denote by L the separable closure of $k(X)$ in $k(\mathbf{P}^1_k)$. The purely inseparable finite extension $k(\mathbf{P}^1_k)/L$, say of degree p^n, corresponds to the n-th iteration of (relative) Frobenius morphism $\mathrm{Frob}^n_{\mathbf{P}^1_k} : \mathbf{P}^1_k \to \mathbf{P}^1_k$ which sends $(x_0 : x_1) \mapsto (x_0^{p^n} : x_1^{p^n})$. So the morphism f can be factorized as

$$\mathbf{P}^1_k \xrightarrow{\mathrm{Frob}^n_{\mathbf{P}^1_k}} \mathbf{P}^1_k \xrightarrow{g} G,$$

where g corresponds to the separable field extension $L/k(X)$. Applying the above argument to g, we still get a contradiction. This completes the proof. □

We end this section by the following proposition due to A. Weil.

Proposition 1.16. *Let X be a normal variety over k, and f be a rational map from X to a group scheme G over k. If f is defined on an open subscheme $U \subset X$ with $\mathrm{codim}_X(X - U) \geq 2$, then f extends to a morphism $X \to G$.*

Proof. We may assume $k = \bar{k}$. Let $U \subset X$ be the maximal open subscheme where f is defined. We write multiplicatively the group law on G. Consider the rational map $\Phi : X \times X \dashrightarrow G$ given by $\Phi(x, y) = f(x)f(y)^{-1}$. We claim that for any $x \in X(k)$, we have $x \in U(k)$ if and only if Φ can be defined at (x, x). The "only if" part is trivial. Now suppose that Φ is defined at (x, x). Let W denote the maximal open locus where Φ is defined, and W_x denote the open subset of X such that $\{x\} \times W_x = W \cap (\{x\} \times X)$. We have $W_x \neq \emptyset$. As X is irreducible, there exists $y \in U \cap W_x$. Thus $f(x) = \Phi(x, y)f(y)$ is well defined. This proves the claim. By assumption, the codimension of $F = X - U$ in X is at least 2. We have to show that Φ is defined everywhere on the diagonal $\Delta(X) \subset X \times X$. We note first that the locus in $\Delta(X)$ where Φ is not defined is exactly $\Delta(F)$, and $\Phi(x, x) = e$ whenever Φ is defined at (x, x), where $e \in G$ denotes the identity element. Let D be the closed subset of $X \times X$ where Φ is not defined. Then each irreducible component of $D \cap \Delta(X)$ must be of codimension 1 in $\Delta(X)$. But by assumption $D \cap \Delta(X) = \Delta(F)$ has codimension at least 2 in $\Delta(X)$. It follows that $D \cap \Delta(X) = \emptyset$. In particular, Φ is defined at (x, x). This completes the proof of the proposition. □

2. Definition and basic properties of Abelian varieties

Definition 2.1. An abelian variety over k is a proper variety over k equipped with a k-group scheme structure.

Proposition 2.2. *Let X be an abelian variety over k.*

(i) *X is smooth over k.*

(ii) *Let $\omega_X = e^*\Omega^1_{X/k}$ be the cotangent space of X at the unit section. Then we have $\Gamma(X, \Omega^1_X) \simeq \omega_X$. In particular, if X has dimension 1, then the genus of X equals 1.*

(iii) *Let Y be a normal variety, and $f : Y \dashrightarrow X$ be a rational map. Then f extends to a morphism $f : Y \to X$.*

(iv) *If Y is a rational variety (*i.e.*, birationally equivalent to the projective space $\mathbf{P}^d_k$ with $d \geq 1$), then any rational map from Y to X is constant.*

Proof. Statement (i) follows from Proposition 1.9(ii). For (ii), it follows from 1.14 that $\Omega^1_{X/k} \simeq \omega_X \otimes_k \mathcal{O}_X$. So we have

$$\Gamma(X, \Omega^1_X) = \omega_X \otimes \Gamma(X, \mathcal{O}_X).$$

But by Lemma 1.8, X is geometrically connected. Hence, we have $\Gamma(X, \mathcal{O}_X) = k$, and (ii) follows. For statement (iii), we note that the local ring of X at a point of height 1 is a discrete valuation ring as X is normal. It follows from the valuative criterion of properness that the rational map f can be defined at all points of height 1. Proposition 1.16 implies that f extends actually to the whole X. For (iv), we note that X is birationally equivalent to $(\mathbf{P}^1_k)^d$, and giving a rational map from Y to X is equivalent to giving a rational from $(\mathbf{P}^1_k)^d$ to X. Statement (iv) now follows immediately from (iii) and Corollary 1.15. □

Proposition 2.3 (Rigidity Lemma)**.** *Let X and Y be varieties over k, Z be a separated k-scheme, and $f : X \times Y \to Z$ be a morphism. Assume that X is proper with a k-rational point, and there exists a closed point $y_0 \in Y$ such that the image $f(X \times \{y_0\})$ is a single point $z_0 \in Z$. Then there is a morphism $g : Y \to Z$ such that $f = g \circ p_2$, where $p_2 : X \times Y \to Y$ is the natural projection.*

Proof. Choose a k-rational point x_0 of X, and define $g : Y \to Z$ by $g(y) = f(x_0, y)$. Since Z is separated, the locus in $X \times Y$ where f and $g \circ p_2$ coincide is closed in $X \times Y$. As $X \times Y$ is connected, to show that $f = g \circ p_2$, we just need to show that these morphisms coincide on some open subset of $X \times Y$. Let U be an affine open neighborhood of z_0 in Z, $F = Z \backslash U$. Then $G = p_2(f^{-1}(F))$ is a closed subset of Y. Since $f(X \times \{y_0\}) = \{z_0\}$ by assumption, we have $y_0 \notin G$. There exists thus an affine open neighborhood V of y_0 such that $V \cap G = \emptyset$. It's easy to see that $f(X \times V) \subset U$. Since U is affine, the morphism $f : X \times V \to U$ is determined by the induced morphism

$$f^* : \Gamma(U, \mathcal{O}_U) \to \Gamma(X \times V, \mathcal{O}_{X \times V}) \simeq \Gamma(X, \mathcal{O}_X) \otimes \Gamma(V, \mathcal{O}_V).$$

As X is proper, reduced, connected and has a k-rational point, we have $\Gamma(X, \mathcal{O}_X) \simeq k$. That means the morphism $f : X \times V \to U$ actually factors through the projection $p_2 : X \times V \to V$. Hence f and $g \circ p_2$ coincide on $X \times V$. □

Corollary 2.4. *Let X be an abelian variety over k, Y be a group scheme over k, and $f : X \to Y$ be a morphism of k-schemes. Then there exists a point $a \in Y(k)$ and a homomorphism of group schemes $h : X \to Y$ such that $f = T_a \circ h$, where T_a is the right translation by a.*

Proof. Let e be the unit section of X, and $a = f(e)$. Define $h : X \to Y$ by $h(x) = f(x) \cdot a^{-1}$. Consider the morphism

$$\Phi : X \times X \to Y \quad (u, v) \mapsto h(uv)h(v)^{-1}h(u)^{-1}.$$

We have $\Phi(e,x) = \Phi(x,e) = e$ for any point x in X. By the rigidity lemma, it follows that Φ is the constant map to e. Hence, h is a homomorphism of abelian varieties. $\square$

Corollary 2.5. *Any abelian variety over k is a commutative group scheme.*

Proof. By Corollary 2.4, any morphism of abelian varieties that sends the unit section to the unit section is a homomorphism. The corollary then follows by applying this fact to the inverse morphism of an abelian variety. $\square$

From now on, we denote additively the group law of an abelian variety X, by 0 its unit element. Let Y and Z be reduced closed subschemes of X. Assume that either Y or Z is geometrically reduced. Denote by $Y+Z$ the image of $Y \times Z$ the addition morphism $m\colon X \times X \to X$, which is a closed subset of X since m is proper. If we endow $Y+Z$ with the reduced closed subscheme structure, then m induces a surjection $Y \times Z \to Y+Z$.

Lemma 2.6. *Let X be an abelian variety over k, and $Y \subset X$ be a closed subvariety stable under the addition morphism. Then Y contains 0 and is stable under the inversion morphism; in particular, Y is an abelian variety.*

Proof. Consider the isomorphism

$$\Phi : X \times X \to X \times X \quad (x,y) \mapsto (x, x+y).$$

Since Y is stable under addition, the image $\Phi(Y \times Y)$ lies in $Y \times Y$. But both $Y \times Y$ and $\Phi(Y \times Y)$ are irreducible varieties of the same dimension. We have $\Phi : Y \times Y \simeq Y \times Y$. In particular, for any $y \in Y$, $\Phi^{-1}(y,y) = (y,0)$ belongs to $Y \times Y$. Thus 0 belongs to Y. Moreover, $\Phi^{-1}(y,0) = (y,-y)$ belongs to $Y \times Y$. This proves Y is stable under inversion. $\square$

Definition 2.7. Let X be an abelian variety over k. We say a closed subvariety $Y \subset X$ is an *abelian subvariety* if Y is stable under addition. We say X is a *simple* abelian variety if it has no non-trivial abelian subvarieties.

Lemma 2.8. *Let X be an abelian variety of dimension d, and W be a geometrically irreducible closed subvariety of X containing 0. Then there exists a unique abelian subvariety $Y \subset X$ containing W such that for any abelian subvariety A of X containing W, we have $Y \subset A$. Moreover, there exists an integer $1 \leq h \leq d$ such that any point $x \in Y(\overline{k})$ can be represented as $\sum_{i=1}^{h} a_i$ with $a_i \in W(\overline{k})$.*

Proof. If $\dim(W) = 0$, then W reduces to $\{0\}$, and the lemma is trivial. Suppose $\dim(W) \geq 1$. For any integer $n \geq 1$, let $W^{(n)}$ be the image of

$$\underbrace{W \times W \times \cdots \times W}_{n \text{ times}} \to X \qquad (x_1, \cdots, x_n) \mapsto x_1 + x_2 + \cdots + x_n.$$

Then $W^{(n)}$ is a closed geometrically irreducible subvariety of X, and we have $W^{(n)} \subset W^{(n+1)}$. It's clear that any abelian subvariety containing W must contain $W^{(n)}$, and any point $x \in W^{(n)}(\overline{k})$ can be written as $\sum_{i=1}^{n} a_i$ with $a_i \in W(\overline{k})$. Let h be the minimal integer such that $W^{(h)} = W^{(h+1)}$. By induction, we see that $W^{(h)} = W^{(n)}$ for any $n \geq h$. As $1 \leq \dim(W^{(m)}) < \dim(W^{(m+1)})$ for $m \leq h-1$, we have $h \leq d$. For $x, y \in W^{(h)}$, we have $x + y \in W^{(2h)} = W^{(h)}$. By 2.6, this means that $Y = W^{(h)}$ is an abelian subvariety of X. $\square$

In the situation of the above lemma, we say Y is the abelian subvariety generated by W.

Proposition 2.9. *Let X be an abelian variety over k of dimension d, D be the support of a divisor of X, W be a closed subvariety containing 0 and disjoint from D, and Y the abelian subvariety generated by W. Then D is stable under translation by Y,* i.e., *$D+Y=D$ in the notations of 2.5.*

Proof. We may assume k algebraically closed. Up to replacing D by one of its irreducible components, we may assume D is irreducible. Let X_1 be the image of the morphism $D\times W\to X$ given by $(x,y)\mapsto x-y$. Then X_1 is an irreducible closed subvariety of X containing D, since $0\in W$. So we have either $X_1=X$ or $X_1=D$. If $X_1=X$, as $0\in X$, we have $0=x-y$ with $x\in D(k)$ and $y\in W(k)$. This means $x=y\in D\cap W$, which contradicts with the assumption that D and W are disjoint. We have thus $X_1=D$, *i.e.,* we have $a-w\in D$ for any $a\in D(k)$ and $w\in W(k)$. Since any $b\in Y(k)$ can be written as $b=-\sum_{i=1}^h w_i$ for some $h\leq d$ and $w_i\in W(k)$ by Lemma 2.8, we see by induction that D contains $D+Y$. □

Corollary 2.10. *Let D be a divisor of an abelian variety X, and W be a closed subvariety of X disjoint from D. Then for any points $w,w'\in W(k)$, D is stable under the translation by $w'-w$.*

Proof. We may assume D effective and reduced. Note that $T_{-w}(W)$ contains 0 and is disjoint from $T_{-w}(D)$. The corollary follows immediately from the proposition. □

Corollary 2.11. *Let X be a simple abelian variety, D be a nontrivial divisor of X. Then any closed subvariety of X of positive dimension has a nontrivial intersection with D.*

3. Theorem of the cube and its consequences

We will assume the following theorem in algebraic geometry, and its proof can be found in [Ha77] or [Mu70, §5].

Theorem 3.1. *Let $f:X\to Y$ be a proper morphism of locally noetherian schemes, $\mathscr{F}$ be a coherent sheaf on X, flat over Y. Then there is a finite complex concentrated in degrees $[0,n]$*

$$K^\bullet : 0\to K^0\to K^1\to\cdots\to K^n\to 0$$

consisting of $\mathcal{O}_Y$-modules locally free of finite type, such that for any morphism $g:Z\to Y$ and any integer $q\geq 0$ we have a functorial isomorphism

$$H^q(X\times_Y Z, p_1^*(\mathscr{F}))\xrightarrow{\sim} H^q(Z, g^*(K^\bullet)),$$

where p_1 is the projection $p_1: X\times_Y Z\to X$.

This important theorem has many consequences on the cohomology of schemes. Here, what we need is the following

Corollary 3.2. *Let X,Y,f and $\mathscr{F}$ be as in the theorem. Then for any integer $q\geq 0$, the function on Y with values in $\mathbf{Z}$ defined by*

$$y\mapsto \dim_{\kappa(y)} H^q(X_y,\mathscr{F}_y)$$

is upper semi-continuous on Y, i.e., *for any integer $d\geq 0$ the subset $\{y\in Y; \dim_{\kappa(y)} H^q(X_y,\mathscr{F}_y)\geq d\}$ is closed in Y.*

Proof. The problem is local for Y, so we may assume $Y=\mathrm{Spec}(A)$ is affine and all the components of the complex $K^\bullet$ are free A-modules of finite type. Let $d^q: K^q\to K^{q+1}$ be the coboundary operator of K. Then we have

$$\begin{aligned}\dim_{\kappa(y)} H^q(X_y,\mathscr{F}_y) &= \dim_{\kappa(y)}\mathrm{Ker}(d^q\otimes\kappa(y)) - \dim_{\kappa(y)}\mathrm{Im}(d^{q-1}\otimes\kappa(y))\\ &= \dim_{\kappa(y)}(K^q\otimes\kappa(y)) - \dim_{\kappa(y)}\mathrm{Im}(d^q\otimes\kappa(y)) - \dim_{\kappa(y)}\mathrm{Im}(d^{q-1}\otimes\kappa(y)).\end{aligned}$$

The first term being constant on Y, it suffices to prove that, for any q, the function $y \mapsto g(y) = \dim_{\kappa(y)} \mathrm{Im}(d^q \otimes_A \kappa(y))$ is lower semi-continuous on Y, *i.e.*, the subset of Y consisting of points y with $g(y) \leq r$ is closed for any integer $r \geq 0$. The condition that $g(y) \leq r$ is equivalent to saying that the morphism $(\wedge^{r+1} d^q) \otimes_A \kappa(y) : K^q \otimes_A \kappa(y) \to K^{q+1} \otimes_A \kappa(y)$ is zero. Since both K^q and K^{q+1} are free A-modules, $\wedge^{r+1} d^q$ is represented by a matrix with coefficients in A. The locus where $\wedge^{r+1} d^q$ vanishes is the common zeros of all the coefficients of its matrix.

□

Proposition 3.3 (See-Saw principle). *Let X be a proper variety, Y be a locally noetherian scheme over k and L be a line bundle on $X \times Y$. Then there exists a unique closed subscheme $Y_1 \hookrightarrow Y$ satisfying the following properties:*

(i) *If L_1 is the restriction of L to $X \times Y_1$, there is a line bundle M_1 on Y_1 and an isomorphism $p_2^* M_1 \simeq L_1$ on $X \times Y_1$;*

(ii) *If $f : Z \to Y$ is a morphism such that there exists a line bundle K on Z and an isomorphism $p_2^*(K) \simeq (\mathrm{Id}_X \times f)^*(L)$ on $X \times Z$, then f can be factored through $g : Z \to Y_1$ and $K \simeq g^*(M_1)$.*

First, we prove the following

Lemma 3.4. *Let X, Y and L be as in the proposition above. Then the subset of Y, consisting of points y such that the restriction L_y on $X \times \{y\}$ is trivial, is closed in Y.*

Proof. We claim that L_y is trivial if and only if we have both $\dim_{\kappa(y)} H^0(X \times \{y\}, L_y) \geq 1$ and $\dim_{\kappa(y)} H^0(X \times \{y\}, L_y^{-1}) \geq 1$. These conditions are clearly necessary. Conversely, if these dimension conditions are satisfied, then there are non-trivial morphisms $f : \mathcal{O}_{X\times\{y\}} \to L_y$ and $g : L_y \to \mathcal{O}_{X\times\{y\}}$. Since X is a proper variety, we have $\Gamma(X \times \{y\}, \mathcal{O}_{X\times\{y\}}) = \kappa(y)$. Hence the composite $g \circ f : \mathcal{O}_{X\times\{y\}} \to L_y \to \mathcal{O}_{X\times\{y\}}$ is necessarily an isomorphism. This shows that both f and g are isomorphisms, hence L_y is trivial. The lemma then follows immediately from Corollary 3.2. □

Proof of Prop. 3.3. The uniqueness of Y_1 follows immediately from the universal property of Y_1. Since different local pieces of Y_1 will patch together by the uniqueness of Y_1, it's sufficient to prove the existence of Y_1 locally for the Zariski topology of Y. Let F be the subset of points $y \in Y$ such that the restriction L_y to $X \times \{y\}$ is trivial. Then F is a closed subset by Lemma 3.4. If the desired Y_1 exists, then its underlying topological space is exactly F. Let $y \in F$ be a closed point, and Y_y be the localization of Y at y. We just need to prove the existence of Y_1 for Y_y, since then Y_1 will naturally spread out to a closed subscheme of a certain open neighborhood of y in Y. Up to replacing Y by Y_y, we may assume $Y = \mathrm{Spec}(A)$ is local with closed point y and L_y is trivial.

Let $K^\bullet = (0 \to K^0 \xrightarrow{\alpha} K^1 \to \cdots)$ be the complex of finite free A-modules given by Theorem 3.1 for the sheaf L. For an A-module N, we put $N^* = \mathrm{Hom}_A(N, A)$. Let M denote the cokernel of the induced map $\alpha^* : K^{1*} \to K^{0*}$. Then for any A-algebra B, if we denote $X_B = X \otimes_k B$ and by L_B the pullback of L on X_B, we have

$$H^0(X_B, L_B) = \mathrm{Ker}(\alpha \otimes_A B) = \mathrm{Hom}_A(M, B) = \mathrm{Hom}_B(M \otimes_A B, B).$$

In particular, we have $H^0(X \times \{y\}, L_y) = \mathrm{Hom}_{\kappa(y)}(M \otimes_A \kappa(y), \kappa(y))$. Since L_y is trivial by assumption, we have $\dim_{\kappa(y)}(M \otimes_A \kappa(y)) = 1$. By Nakayama's lemma, there exists an ideal $I \subset A$ such that $M \simeq A/I$ as A-modules. Then for any A-algebra B, $H^0(X_B, L_B)$ is a free B-module of rank 1 if and only if the structure map $A \to B$ factors as $A \to A/I \to B$. Applying the same process to the sheaf L^{-1}, we get another ideal $J \subset A$ such that for any A-algebra B, $H^0(X_B, L_B^{-1})$ is free of rank 1 over B if and only if B is actually an A/J-algebra. We claim that the closed subscheme $Y_1 = \mathrm{Spec}(A/(I + J))$ satisfies the requirements of the proposition. Condition (ii) is immediate by our construction of $X \times Y_1$. Let L_1 denote the restriction of L on Y_1. It's sufficient to prove that L_1 is a trivial line bundle. Let f_0 and g_0 be respectively generators of $H^0(X \times \{y\}, L_y)$ and $H^0(X \times \{y\}, L_y^{-1})$ such that their image is 1 by the canonical map

$$H^0(X \times \{y\}, L_y) \times H^0(X \times \{y\}, L_y^{-1}) \to H^0(X \times \{y\}, \mathcal{O}_{X \times \{y\}}) = \kappa(y).$$

This is certainly possible, since L_y is trivial. Let $f \in H^0(X \times Y_1, L_1)$ (*resp.* $g \in H^0(X \times Y_1, L_1^{-1})$) be a lift of f_0 (*resp.* g_0). Up to modifying f, we may assume the image of (f, g) is 1 by the canonical product morphism

$$H^0(X \times Y_1, L_1) \times H^0(X \times Y_1, L_1^{-1}) \to H^0(X \times Y_1, \mathcal{O}_{X \times Y_1}) \simeq A/(I + J).$$

If we denote still by $f : \mathcal{O}_{X \times Y_1} \to L_1$ (*resp.* by $g : L_1 \to \mathcal{O}_{X \times Y_1}$) the morphism of line bundles induced by f (*resp.* by g), we have $g \circ f = \mathrm{Id}_{\mathcal{O}_{X \times Y_1}}$ and $f \circ g = \mathrm{Id}_{L_1}$. This proves that L_1 is trivial. □

Theorem 3.5 (Theorem of the Cube). *Let X and Y be proper varieties over k, Z be a connected k-scheme of finite type, and $x_0 \in X(k)$, $y_0 \in Y(k)$ and $z_0 \in Z$. Let L be a line bundle on $X \times Y \times Z$ whose restrictions to $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ are trivial. Then L is trivial.*

Proof. (Mumford) By Proposition 3.3, there exists a maximal closed subscheme $Z' \subset Z$ such that $L|_{X \times Y \times Z'} \simeq p_3^*(M)$, where p_3 is the projection $X \times Y \times Z' \to Z'$ and M is a line bundle on Z'. As $z_0 \in Z'$, Z' is non-empty. After restriction to $\{x_0\} \times Y \times Z'$, we see that $M \simeq \mathcal{O}'_Z$. It remains to show that $Z' = Z$. Since Z is connected, it suffices to prove that if a point belongs to Z', then Z' contains an open neighborhood of this point. Denote this point by z_0. Let $\mathcal{O}_{Z,z_0}$ be the local ring of Z at z_0, $\mathfrak{m}$ be its maximal ideal, and $\kappa(z_0) = \mathcal{O}_{Z,z_0}/\mathfrak{m}$, and I_{z_0} be the ideal of $Z' \times_Z \mathrm{Spec}(\mathcal{O}_{Z,z_0})$. It's sufficient to prove that $I_{z_0} = 0$. If not, since $\cap_{n \geq 1} \mathfrak{m}^n = 0$ by Krull's theorem, we would have an integer $n \geq 1$ such that $\mathfrak{m}^n \supset I_{z_0}$ and $\mathfrak{m}^{n+1} \not\supseteq I_{z_0}$. Hence $(\mathfrak{m}^{n+1} + I_{z_0})/\mathfrak{m}^{n+1}$ is a non-zero subspace of $\mathfrak{m}^n/\mathfrak{m}^{n+1}$. We put $J_1 = \mathfrak{m}^{n+1} + I_{z_0}$, then there exists $\mathfrak{m}^{n+1} \subset J_2 \subsetneq J_1$ such that $\dim_{\kappa(z_0)}(J_1/J_2) = 1$. Let $Z_i = \mathrm{Spec}(\mathcal{O}_{Z,z_0}/J_i)$ for $i = 1, 2$. We have $Z_1 \subset Z_2$, and the ideal of Z_1 in Z_2 is generated by an element $a \in I_{z_0}$. We have an exact sequence of abelian sheaves over the topological space $X \times Y \times \{z_0\}$

$$0 \to \mathcal{O}_{X \times Y \times \{z_0\}} \xrightarrow{u} \mathcal{O}^{\times}_{X \times Y \times Z_2} \to \mathcal{O}^{\times}_{X \times Y \times Z_1} \to 1,$$

where u is given by $x \mapsto 1 + ax$. Since $H^0(X \times Y \times Z_i, \mathcal{O}^{\times}_{X \times Y \times Z_i})$ is canonically isomorphic to $H^0(Z_i, \mathcal{O}^{\times}_{Z_i})$ for $i = 1, 2$, we see that the natural map $H^0(X \times Y \times Z_2, \mathcal{O}^{\times}_{X \times Y \times Z_2}) \to H^0(X \times Y \times Z_1, \mathcal{O}^{\times}_{X \times Y \times Z_1})$ is surjective. Hence, we have an exact sequence of cohomology groups

(3.5.1)

$$0 \to H^1(X \times Y \times \{z_0\}, \mathcal{O}_{X \times Y \times \{z_0\}}) \to H^1(X \times Y \times \{z_0\}, \mathcal{O}^{\times}_{X \times Y \times Z_2}) \to H^1(X \times Y \times \{z_0\}, \mathcal{O}^{\times}_{X \times Y \times Z_1}).$$

By our construction, $L_1 = L|_{X \times Y \times Z_1}$ is trivial, and $L_2 = L|_{X \times Y \times Z_2}$ is not trivial. If we denote by $[L_i]$ $(i = 1, 2)$ the cohomology class of L_i in $H^1(X \times Y \times \{z_0\}, \mathcal{O}^{\times}_{X \times Y \times Z_i})$, we have $[L_1] = 0$ and $[L_2] \neq 0$. By the exact sequence (3.5.1), $[L_2]$ comes from a nonzero cohomology class $[L_0]$ in $H^1(X \times Y \times \{z_0\}, \mathcal{O}_{X \times Y \times \{z_0\}})$. We have a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1(X \times Y \times \{z_0\}, \mathcal{O}_{X \times Y \times \{z_0\}}) & \longrightarrow & H^1(X \times Y \times \{z_0\}, \mathcal{O}^{\times}_{X \times Y \times Z_2}) \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(\{x_0\} \times Y \times \{z_0\}, \mathcal{O}_{\{x_0\} \times Y \times \{z_0\}}) & \longrightarrow & H^1(\{x_0\} \times Y \times \{z_0\}, \mathcal{O}^{\times}_{\{x_0\} \times Y \times Z_2}), \end{array}$$

where vertical arrows are induced by the natural restriction $\{x_0\} \times Y \times \{z_0\} \hookrightarrow X \times Y \times \{z_0\}$, and the injectivity of the lower arrow follows in the same way as in (3.5.1). Since L is trivial over $\{x_0\} \times Y \times Z$, in particular over $\{x_0\} \times Y \times Z_2$, the image of $[L_0]$ in

$$H^1(\{x_0\} \times Y \times \{z_0\}, \mathcal{O}_{\{x_0\}\times Y\times\{z_0\}}) \simeq H^1(Y, \mathcal{O}_Y) \otimes_k \kappa(z_0)$$

vanishes. Similarly, the image of $[L_0]$ in $H^1(X \times \{y_0\} \times \{z_0\}, \mathcal{O}_{X\times\{y_0\}\times\{z_0\}}) \simeq H^1(X, \mathcal{O}_X) \otimes_k \kappa(z_0)$ vanishes. On the other hand, we have an isomorphism

$$H^1(X\times Y\times\{z_0\}, \mathcal{O}_{X\times Y\times\{z_0\}}) \simeq H^1(X\times Y, \mathcal{O}_{X\times Y})\otimes_k \kappa(z_0) \xrightarrow{\sim} (H^1(X, \mathcal{O}_X)\oplus H^1(Y, \mathcal{O}_Y))\otimes_k \kappa(z_0)$$

by Künneth formula, where the latter map is induced by the inclusion $(\{x_0\} \times Y) \cup (X \times \{y_0\}) \hookrightarrow X \times Y$. We conclude that $[L_0]$ must vanish. This is a contradiction, and the proof of the theorem is complete.

□

Remark 3.6. A slightly different way to prove that Z' contains an open neighborhood of z_0 is the following. First, we note as above that it's sufficient to show the restriction of L to $X \times Y \times \mathrm{Spec}(\mathcal{O}_{Z,z_0})$ is trivial. Let A be the completion of $\mathcal{O}_{Z,z_0}$, $S = \mathrm{Spec}(A)$, $S_n = \mathrm{Spec}(\mathcal{O}_{Z,z_0}/\mathfrak{m}^{n+1})$, and $(X \times Y \times S)^\wedge$ be the completion of $X \times Y \times S$ along the closed subscheme $X \times Y \times \{z_0\}$. Then we have canonical morphisms of Picard groups

$$\mathrm{Pic}(X \times Y \times \mathrm{Spec}(\mathcal{O}_{Z,z_0})) \hookrightarrow \mathrm{Pic}(X \times Y \times S) \xrightarrow{\sim} \mathrm{Pic}((X \times Y \times S)^\wedge) = \varprojlim_n \mathrm{Pic}(X \times Y \times S_n),$$

where the injectivity of this map follows from the descent of coherent sheaves by faithfully flat and quasi-compact morphisms, and the second isomorphism is Grothendieck's existence theorem of coherent sheaves in formal geometry [EGA III 5.1.4]. Hence it suffices to prove that the restriction $L_n = L|_{X\times Y\times S_n}$ is trivial for all $n \geq 0$. We prove this by induction on n. The case $n = 0$ is a hypothesis of the theorem. We now assume $n \geq 1$ and L_{n-1} is trivial. We have an exact sequence of abelian sheaves on $X \times Y \times \{z_0\}$

$$0 \to \mathcal{O}_{X\times Y\times\{z_0\}} \otimes_k \mathfrak{m}^n/\mathfrak{m}^{n+1} \xrightarrow{u} \mathcal{O}^\times_{X\times Y\times S_n} \to \mathcal{O}^\times_{X\times Y\times S_{n-1}} \to 1,$$

where u is given by $x \mapsto 1 + x$. Taking cohomologies, we get

$$0 \to H^1(X \times Y \times \{z_0\}, \mathcal{O}_{X\times Y\times\{z_0\}}) \otimes_k \mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathrm{Pic}(X \times Y \times S_n) \to \mathrm{Pic}(X \times Y \times S_{n-1}).$$

By induction hypothesis the class of L_{n-1} in $\mathrm{Pic}(X \times Y \times S_{n-1})$ is zero, so the class of L_n in $\mathrm{Pic}(X \times Y \times S_n)$ comes from a class in $H^1(X \times Y \times \{z_0\}, \mathcal{O}_{X\times Y\times\{z_0\}})$. Then we can use the same argument as above to conclude that this cohomology class must vanish.

Proposition 3.7. *Let X be an abelian variety over k, $p_i : X \times X \times X \to X$ be the projection onto the i-th factor, $m_{i,j} = p_i + p_j : X \times X \times X \to X$, and $m_{123} : p_1 + p_2 + p_3 : X \times X \times X \to X$. Then for any line bundle L on X, we have*

$$M = m_{123}^* L \otimes m_{12}^* L^{-1} \otimes m_{13}^* L^{-1} \otimes m_{23}^* L^{-1} \otimes p_1^* L \otimes p_2^* L \otimes p_3^* L \simeq \mathcal{O}_{X\times X\times X}.$$

Equivalently, if S is a k-scheme and f, g, h are any S valued points of X, we have

(3.7.1) $$(f + g + h)^* L \simeq (f + g)^* L \otimes (f + h)^* L \otimes (g + h)^* L \otimes f^* L^{-1} \otimes g^* L^{-1} \otimes h^* L^{-1}.$$

Proof. Let $i_1 : X \times X \to X \times X \times X$ be the morphism given by $(x, y) \mapsto (0, x, y)$. We have $m_{123} \circ i_1 = m$, $m_{12} \circ i_1 = p_1$, $m_{13} \circ i_1 = p_2$, $m_{23} \circ i_1 = m$, $p_1 \circ i_1 = 0$, $p_2 \circ i_1 = p_1$, and $p_3 \circ i_1 = p_2$. So we have

$$M|_{\{0\}\times X\times X} = i_1^* M = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \otimes m^* L^{-1} \otimes p_1^* L \otimes p_2^* L \simeq \mathcal{O}_{X\times X}.$$

Similarly, both $M|_{X\times\{0\}\times X}$ and $M_{X\times X\times\{0\}}$ are trivial. The corollary follows from Theorem 3.5. $\square$

Corollary 3.8. *Let X be an abelian variety, n_X be the morphism of multiplication by $n\in\mathbf{Z}$ on X. Then for any line bundle L on X, we have*

$$(n_X)^*L\simeq L^{\frac{n^2+n}{2}}\otimes(-1_X)^*L^{\frac{n^2-n}{2}}. \tag{3.8.1}$$

Proof. The formula (3.8.1) for $n<0$ follows from the case $n>0$ by applying $(-1)^*_X$. We now prove the corollary for $n\geq 1$ by induction. The cases with $n=0,1$ are trivial. Assume now $n\geq 1$ and (3.8.1) has been verified for all positive integers less than or equal to n. Taking $f=n_X$, $g=1_X$, and $h=-1_X$ in the formula (3.7.1), we get

$$(n+1)^*_XL\simeq(n^*_XL)^2\otimes(n-1)^*_XL^{-1}\otimes L\otimes(-1_X)^*L.$$

The formula (3.8.1) is then verified by an easy computation. $\square$

Corollary 3.9 (Square Theorem)**.** *Let X be an abelian variety over k, L a line bundle on X. Let S be any k-scheme, $X_S=X\times S$, $L_S=p_X^*L$, x,y be two S-valued points of X, and $T_x:X_S\to X_S$ be the translation by x. Then there exists a line bundle N on S such that*

$$T^*_{x+y}L_S\otimes L_S\simeq T^*_xL_S\otimes T^*_yL_S\otimes p^*_SN,$$

where $p_S:X_S\to S$ is the natural projection onto S.

Proof. Let $\beta:S\to X\times X$ be the morphism $s\mapsto(x(s),y(s))$, and $\alpha=(\mathrm{Id}_X,\beta):X_S=X\times S\to X\times(X\times X)$. By Theorem of the cube 3.7, we have

$$\alpha^*(m^*_{123}L)\simeq\alpha^*(m^*_{12}L\otimes m^*_{13}L\otimes m^*_{23}L\otimes p^*_1L^{-1}\otimes p^*_2L^{-1}\otimes p^*_3L^{-1})$$

It's easy to see that the above isomorphism is equivalent to

$$T^*_{x+y}L_S=T^*_xL_S\otimes T^*_yL_S\otimes L_S^{-1}\otimes p^*_S(m^*L\otimes p_1^{-1}L\otimes p_2^{-1}L).$$

This proves the corollary. $\square$

Definition 3.10. Let X be an abelian variety, and L a line bundle on X. We denote by $K(L)$ the maximal closed subscheme X such that $(m^*(L)\otimes p^*_1L^{-1})|_{X\times K(L)}$ has the form $p^*_2(N)$.

The existence of $K(L)$ is ensured by see-saw principle 3.4. It's easy to see that $0\in K(L)$. Restricted to $\{0\}\times K(L)$, we have $N\simeq L$. Thus $K(L)$ is the maximal subscheme $Z\subset X$ such that the restriction of the line bundle $(m^*L\otimes p^*_1L^{-1}\otimes p^*_2L^{-1})$ to $X\times Z$ is trivial.

Lemma 3.11. *The closed subscheme $K(L)$ is a closed subgroup scheme of X.*

Proof. We have to show that $K(L)$ is stable under the addition of X, *i.e.,* it's sufficient to prove that the morphism

$$X\times K(L)\times K(L)\hookrightarrow X\times X\times X\xrightarrow{\mathrm{Id}_X\times m}X\times X$$

factors through the natural inclusion $X\times K(L)\hookrightarrow X\times X$. By the universal property of $K(L)$, we need to prove that the restriction of $(\mathrm{Id}_X\times m)^*(m^*L\otimes p^*_1L^{-1}\otimes p^*_2L^{-1})$ to $X\times K(L)\times K(L)$ is trivial. With the notation of 3.7, we have $m\circ(\mathrm{Id}_X\times m)=m_{123}$, $p_1\circ(\mathrm{Id}_X\times m)=p_1$, and $p_2\circ(\mathrm{Id}_X\times m)=m_{23}$. Therefore, we have

$$\begin{aligned}(\mathrm{Id}_X\times m)^*(m^*L\otimes p^*_1L^{-1}\otimes p^*_2L^{-1})&=m^*_{123}L\otimes p^*_1L^{-1}\otimes m^*_{23}L^{-1}\\&\simeq m^*_{12}L\otimes m^*_{13}L\otimes p^*_1L^{-2}\otimes p^*_2L^{-1}\otimes p^*_3L^{-1}\\&=(m^*_{12}L\otimes p^*_1L^{-1}\otimes p^*_2L^{-1})\otimes(m^*_{13}L\otimes p^*_1L^{-1}\otimes p^*_3L^{-1}),\end{aligned}$$

where the second isomorphism uses Proposition 3.7. When restricted to $X \times K(L) \times K(L)$, both $m_{12}^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$ and $m_{13}^*L \otimes p_1^*L^{-1} \otimes p_3^*L^{-1}$ are trivial. This proves the lemma. □

Proposition 3.12. *Consider the statements:*
(i) *L is ample on X.*
(ii) *$K(L)$ is a finite group scheme.*
We have (i) $\Rightarrow$ (ii). *Conversely, if $L = \mathcal{O}_X(D)$ is the line bundle associated with an effective divisor D, then* (i) *and* (ii) *are both equivalent to*
(iii) *The group $\{x \in X(\overline{k}); T_x(D) = D\}$ (equality as divisors) is finite.*

Proof. By considering scalar extension of X to $\overline{k}$, we may assume that k is algebraically closed. For (i)$\Rightarrow$(ii), suppose $K(L)$ is not finite. Let Y be the reduced closed subscheme of the neutral connected component of $K(L)$. Then Y is a smooth connected closed subgroup scheme of X of dimension $d > 0$; in particular, Y is an abelian scheme. As L is ample, so is the restriction $L_Y = L|_Y$. By our construction, the line bundle $m^*L_Y \otimes p_1^*L_Y^{-1} \otimes p_2^*L_Y^{-1}$ is trivial. Pulling back by the morphism $Y \to Y \times Y$ given by $y \mapsto (y, -y)$, we see that $L_Y \otimes (-1_Y)^*L_Y$ is trivial. But -1_Y is an automorphism of Y and L_Y is ample, so $L_Y \otimes (-1_Y)^*L_Y$ is also ample. This is a contradiction.

Assume $L = \mathcal{O}_X(D)$ is effective. The implication (ii)$\Rightarrow$(iii) being trivial, we just need to prove (iii)$\Rightarrow$(i). By Serre's cohomological criterion of ampleness [EGA, III 4.4.2], it's sufficient to prove that the linear system $|2D|$ has no base point, and defines a finite morphism $X \to \mathbf{P}^N$. By 3.9, the linear system $|2D|$ contains the divisors $T_x(D)+T_{-x}(D)$ (addition for divisors) for all $x \in X(k)$. For any $u \in X(k)$, we can find $x \in X(k)$ such that $u \pm x \notin \operatorname{Supp} D$, *i.e.,* we have $u \notin T_x(D) + T_{-x}(D)$. This shows that the linear system $|2D|$ has no base point, and defines a morphism $\phi : X \to \mathbf{P}^N$. If ϕ is not finite, we can find a closed irreducible curve C such that $\phi(C)$ is a point. It follows that for any $E \in |2D|$, either E contains C or is disjoint with C. In particular, there are infinitely many $x \in X(k)$ such that $T_x(D) + T_{-x}(D)$ is disjoint from C. For such an $x \in X(k)$, it follows from Corollary 2.10 that every irreducible component of $T_x(D) + T_{-x}(D)$ is invariant under the translation by $a - b$ for any $a, b \in C(k)$. In particular, D is invariant under translation by $a - b$ for any $a, b \in C(k)$. This contradicts the assumption that the group $\{x \in X(k); T_x(D) \simeq D\}$ is finite. □

Theorem 3.13. *Every abelian variety over k is projective.*

Proof. Let X be an abelian variety over k of dimension $d \geq 1$. We need to prove that there exists an ample line bundle on X. First, we prove this in the case $k = \overline{k}$. Let U be an affine open subscheme of X such that each irreducible component of the complementary $X \backslash U$ has dimension $d - 1$. We denote by D the divisor defined by the reduced closed subscheme $X \backslash U$. Up to translation, we may assume that $0 \in U$. Consider the subgroup $H \subset X$ defined by

$$H(\overline{k}) = \{x \in X(\overline{k}); T_x(D) = D\}.$$

It's easy to see that H is closed in X. On the other hand, U is stable under translation T_x for $x \in H$. Since $0 \in U$, we have $H \subset U$. It follows that H is both proper and affine, hence finite. The above proposition now implies that $\mathcal{O}_X(D)$ is ample. In the general case, we can choose an ample divisor D defined over a finite extension k'/k. If k'/k is separable, we may assume k'/k is Galois. The divisor

$$\tilde{D} := \sum_{\sigma \in \mathrm{Gal}(k'/k)} \sigma(D)$$

is then an ample divisor defined over k. If k'/k is purely inseparable, there exists an integer m such that $\alpha^{p^m} \in k$ for all $\alpha \in k'$. Then $p^m \cdot D$ is an ample divisor defined on k. The general case is a composition of these two special cases. □

4. Quotient by a finite group scheme

Definition 4.1. Let G be a group scheme over k, $e \in G(k)$ be the unit element, and m denote its multiplication. An action of G on a k-scheme X is a morphism $\mu : G \times X \to X$ such that

(i) the composite

$$X \simeq \mathrm{Spec}(k) \times X \xrightarrow{e \times \mathrm{Id}_X} G \times X \xrightarrow{\mu} X$$

is the identity;

(ii) the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \mathrm{Id}_X} & G \times X \\ \downarrow{\scriptstyle \mathrm{Id}_X \times \mu} & & \downarrow{\scriptstyle \mu} \\ G \times X & \xrightarrow{\mu} & X \end{array}$$

is commutative.

The action μ is said to be free if the morphism

$$(\mu, p_2) : G \times X \to X \times X$$

is a closed immersion.

From the functorial point of view, giving an action of G on X is equivalent to giving for every S-valued point f of G, an automorphism $\mu_f : X \times S \to X \times S$ of S-schemes, functorially in S. The two conditions in the definition above is equivalent to requiring that $G(S) \to \mathrm{Aut}_S(X \times S)$ sending $f \mapsto \mu_f$ is a homomorphism of groups.

Definition 4.2. Let X and Y be k-schemes equipped with a G-action. A morphism of k-schemes $f : Y \to X$ is called *G-equivariant*, if the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{\mu_X} & X \\ \downarrow{\scriptstyle \mathrm{Id}_G \times f} & & \downarrow{\scriptstyle f} \\ G \times Y & \xrightarrow{\mu_Y} & Y \end{array}$$

is commutative. In particular, if the action of G on Y is trivial, a G-equivariant morphism $f : X \to Y$ is called *G-invariant.* A G-invariant morphism $f : X \to \mathbf{A}^1_k$ is called a G-invariant function on X.

Definition 4.3. Let X be a k-scheme equipped with an action of G, $\mathscr{F}$ be a coherent sheaf on X. A *lift* of the action μ to $\mathscr{F}$ is an isomorphism $\lambda : p_2^*\mathscr{F} \xrightarrow{\sim} \mu^*\mathscr{F}$ of sheaves on $G \times X$ such that the diagram of sheaves on $G \times G \times X$

$$\begin{array}{ccccc} p_3^*\mathscr{F} & \xrightarrow{p_{23}^*(\lambda)} & p_{23}^*(\mu^*\mathscr{F}) & = & (\mathrm{Id}_G \times \mu)^*(p_2^*\mathscr{F}) \\ \| & & & & \downarrow{\scriptstyle (\mathrm{Id}_G \times \mu)^*(\lambda)} \\ (m \times \mathrm{Id}_X)^*(p_2^*\mathscr{F}) & \xrightarrow{(m \times \mathrm{Id}_X)^*(\lambda)} & (m \times \mathrm{Id}_X)^*(\mu^*\mathscr{F}) & = & (\mathrm{Id}_G \times \mu)^*(\mu^*\mathscr{F}) \end{array} \tag{4.3.1}$$

is commutative, where $p_{23} : G \times G \times X \to G \times X$ is the natural projection onto the last two factors, p_i is the projection onto the i-th factor.

From the functorial point of view, a lift of μ on $\mathscr{F}$ is to require, for every S-valued point f of G, an isomorphism of coherent sheaves

$$\lambda_f : \mathscr{F} \otimes \mathcal{O}_S \xrightarrow{\sim} \mu_f^*(\mathscr{F} \otimes \mathcal{O}_S)$$

on $X \times S$, where $\mu_f : X \times S \to X \times S$ is the automorphism determined by μ. The commutative diagram (4.3.1) is equivalent to requiring that, for any S-valued points f, g of G, we have a commutative diagram of coherent sheaves on $X \times S$

$$\begin{array}{ccc} \mathscr{F} \otimes \mathcal{O}_S & \xrightarrow{\lambda_g} & \mu_g^*(\mathscr{F} \otimes \mathcal{O}_S) \\ \downarrow{\scriptstyle \lambda_{fg}} & & \downarrow{\scriptstyle \mu_g^*(\lambda_f)} \\ \mu_{fg}^*(\mathscr{F} \otimes S) & = & \mu_g^*\mu_f^*(\mathscr{F} \otimes \mathcal{O}_S), \end{array}$$

where we have used the identification $\mu_{fg} = \mu_f \circ \mu_g$.

Theorem 4.4. *Let G be a finite group scheme over k, X be a k-scheme endowed with an action of G such that the orbit of any point of X is contained in an affine open subset of X.*

(i) *There exists a pair (Y, π) where Y is a k-scheme and $\pi : X \to Y$ a morphism, satisfying the following conditions:*

- *as a topological space, (Y, π) is the quotient of X for the action of the underlying finite group;*
- *the morphism $\pi : X \to Y$ is G-invariant, and if $\pi_*(\mathcal{O}_X)^G$ denotes the subsheaf of $\pi_*(\mathcal{O}_X)$ of G-invariant functions, the natural homomorphism $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism.*

The pair (Y, π) is uniquely determined up to isomorphism by these conditions. The morphism π is finite and surjective. Furthermore, Y has the following universal property: for any G-invariant morphism $f : X \to Z$, there exists a unique morphism $g : Y \to Z$ such that $f = g \circ \pi$.

(ii) *Suppose that the action of G on X is free and $G = \mathrm{Spec}(R)$ with $\dim_k(R) = n$. The π is a finite flat morphism of degree n, and the subscheme of $X \times X$ defined by the closed immersion*

$$(\mu, p_2) : G \times X \to (X, X)$$

is equal to the subscheme $X \times_Y X \subset X \times X$. Finally, if $\mathscr{F}$ is a coherent sheaf on Y, $\pi^*\mathscr{F}$ has a natural G-action lifting that on X, and

$$\mathscr{F} \mapsto \pi^*\mathscr{F}$$

induces an equivalence of the category of coherent $\mathcal{O}_Y$-modules (resp. *locally free $\mathcal{O}_Y$-module of finite rank*) *and the category of coherent $\mathcal{O}_X$-modules with G-action* (resp. *locally free $\mathcal{O}_X$-modules of finite rank with G-action*).

Proof. (i) Let x be a point of X, and U be an affine neighborhood of x that contains the orbit of x. We put $V = \cap_{g \in G(\bar{k})} gU$, where g runs through the set of geometric points of G. Then V is an affine neighborhood of x in X invariant under the action of G. Up to replacing X by V, we may and do assume that $X = \mathrm{Spec}(A)$ is affine. Let R be the ring of G, $\epsilon : R \to k$ the evaluation map at the unit element e, and $m^* : R \to R \otimes R$ be the comultiplication map. Then giving an action μ of G on X is equivalent to giving a map of k-algebras $\mu^* : A \to R \otimes_k A$ such that the composite

$$(\epsilon \otimes \mathrm{Id}_A) \circ \mu^* : A \xrightarrow{\mu^*} R \otimes A \xrightarrow{\epsilon \otimes \mathrm{Id}_A} A$$

is the identity map, and that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mu^*} & R \otimes A \\ \downarrow{\scriptstyle \mu^*} & & \downarrow{\scriptstyle m^* \otimes \mathrm{Id}_A} \\ R \otimes A & \xrightarrow{\mathrm{Id}_R \otimes \mu^*} & R \otimes R \otimes A \end{array}$$

is commutative.

□

5. The Picard functor

Let S be a scheme, $\mathbf{Sch}/S$ be the category of S-schemes, and $f : X \to S$ be a proper and flat morphism. For $T \in \mathbf{Sch}/S$, we put $X_T = X \times_S T$. We denote by $\mathrm{Pic}(X_T)/\mathrm{Pic}(T)$ the cokernel of the natural homomorphism of groups $f_T^* : \mathrm{Pic}(T) \to \mathrm{Pic}(X_T)$ induced by the canonical projection $f_T : X_T \to T$. We denote by $\underline{\mathrm{Pic}}_{X/S}$ be the abelian functor $T \mapsto \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$ on $\mathbf{Sch}/S$. Let $\mathrm{Pic}_{X/S}$ be the associated sheaf of $\underline{\mathrm{Pic}}_{X/S}$ with respect to the fppf-topology on $\mathbf{Sch}/S$. In general, for $T \in \mathbf{Sch}/S$, $\mathrm{Pic}_{X/S}(T)$ does not coincide with $\mathrm{Pic}(X_T)/\mathrm{Pic}(T)$. But we have the following

Lemma 5.1. *If $f : X \to S$ admits a section s, then $\underline{\mathrm{Pic}}_{X/S}$ is a sheaf for the* fppf*-topology of* $\mathbf{Sch}/S$, i.e., *for any $T \in \mathbf{Sch}/S$, we have* $\mathrm{Pic}_{X/S}(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$.

Proof. Let $p : T' \to T$ be a fppf-morphism in $\mathbf{Sch}/S$, $T'' = T' \times_T T'$, and $p_i : T'' \to T'$ be the canonical projections onto the i-th factor. We have to verify that there is an exact sequence of abelian groups

$$0 \to \mathrm{Pic}(X_T)/\mathrm{Pic}(T) \xrightarrow{p^*} \mathrm{Pic}(X_{T'})/\mathrm{Pic}(T') \xrightarrow{p_2^* - p_1^*} \mathrm{Pic}(X_{T''})/\mathrm{Pic}(T'').$$

□

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