Tsinghua 2010 **Abelian Varieties** Problem Set 1

Let k be a field.

1. Let Set be the category of sets, Alg_k be the category of k-algebras, Sch_k be the category of k-schemes, $\mathrm{Sch}_k^{\wedge} = \mathrm{Funct}(\mathrm{Sch}_k^{\mathrm{op}}, \mathrm{Set})$ be the category of contravariant functors from Sch_k to Set. For $F \in \mathrm{Sch}_k^{\wedge}$ and $X \in \mathrm{Sch}_k$, we denote by F_X the presheaf of sets on X given by $F_X(U) = F(U)$ for any open subset U of X. We say F is a *Zariski sheaf* if F_X is a sheaf for all $X \in \mathrm{Sch}_k$. Let $\mathrm{Sch}_k^{\sim}$ be the full subcategory of $\mathrm{Sch}_k^{\wedge}$ consisting of Zariski sheaves.

(a) Show that the fully faithful functor $h\colon \mathrm{Sch}_k \to \mathrm{Sch}_k^{\wedge}$ sending X to $h_X = \mathrm{Mor}(-, X)$ factors through $\mathrm{Sch}_k^{\sim}$.

(b) Let $r\colon \mathrm{Sch}_k^{\wedge} \to \mathrm{Funct}(\mathrm{Alg}_k, \mathrm{Set})$ be the functor induced by the fully faithful functor $\mathrm{Alg}_k^{\mathrm{op}} \to \mathrm{Sch}_k$ given by $A \mapsto \mathrm{Spec}(A)$. Show that the restriction of r to $\mathrm{Sch}_k^{\sim}$ is fully faithful. Deduce that the functor $rh\colon \mathrm{Sch}_k \to \mathrm{Funct}(\mathrm{Alg}_k, \mathrm{Set})$ sending X to $A \mapsto X(A) = \mathrm{Mor}(\mathrm{Spec}(A), X)$ is fully faithful.

(c) Let Gr be the category of groups, GrSch_k be the category of group schemes over k. Show that rh induces a fully faithful functor $\mathrm{GrSch}_k \to \mathrm{Funct}(\mathrm{Alg}_k, \mathrm{Gr})$.

2. Let $f\colon G \to H$ be a homomorphism of k-group schemes. Define the kernel $K = \mathrm{Ker}(f)$ and identify $K(S)$ for all k-scheme S.

3. Assume $\mathrm{char}(k) = p > 0$.

(a) Let G be a k-group scheme. Show that the relative Frobenius $F^n_{G/k}\colon G \to G^{(p^n)}$ is a homomorphism. Describe $G_n = \mathrm{Ker}(F^n_{G/k})$.

(b) Compute $\alpha_{p^n} = (\mathbf{G}_a)_n$ and $(\mathbf{G}_m)_n$.

4. Let $A \subset U$, $B \subset V$ be k-vector spaces, $u \in U$, $v \in V$. Assume that $u \otimes v$ belongs to $A \otimes V + U \otimes B$. Show that either u belongs to A or v belongs to B. (Used in Oort's proof of Cartier's theorem.)

5. Compute Ω^1_G and the space of invariant 1-forms for $G = \mathbf{G}_a, \mathbf{G}_m, \alpha_{p^n}, \mu_n$. Assume $\mathrm{char}(k) = p > 0$ for $G = \alpha_{p^n}$.

6. Let X be a reduced scheme, Y be a separated scheme, $f_1, f_2\colon X \to Y$ be morphisms. Assume that there exists an dense open subset U of X such that $f_1|U = f_2|U$. Show that $f_1 = f_2$.

7. Let R be a ring and M be an R-module. We say that a prime ideal $\mathfrak{p}$ is *associated to* M if $\mathfrak{p}$ is the annihilator of an element of M. We say that a Noetherian scheme satisfies *condition S2* if $\mathrm{depth}(\mathcal{O}_{X,x}) \geq \min\{\dim(\mathcal{O}_{X,x}), 2\}$ for every point x of X.

(a) Let P be the set of annihilators of the nonzero elements of M. Show that every maximal element of P is a prime ideal.

(b) Assume that R is reduced and Noetherian. Let K be the fraction ring of R. Show that an element x of K belongs to R if and only if, for every element u of R which is not a zero divisor, the image of x in $K_{\mathfrak{p}}$ belongs to $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R associated to $R/(u)$.

(c) Let X be a reduced Noetherian scheme satisfying condition S2, Y be an affine scheme. Let U be an open subset of X such that $\mathrm{codim}_X(X - U) \geq 2$, $f\colon U \to Y$ be a morphism. Show that f can be uniquely extended to a morphism $X \to Y$.