Tsinghua 2010 **Abelian Varieties** Problem Set 4

Let k be a field. A variety over k is a geometrically integral and separated scheme of finite type over k.

1. Let X be a proper variety over k with a rational point $x \in X(k)$, and $\underline{\mathrm{Pic}}_{X/k}$ be the Picard functor of X relative to k.

(i) Let $S = \mathrm{Spec}(k[\epsilon]/\epsilon^2)$, $e : \mathrm{Spec}(k) \to S$ be the zero section given by $e^* : \epsilon \mapsto 0$. We define the tangent space of the fuctor $\underline{\mathrm{Pic}}_{X/k}$ at 0 as

$$T_{\underline{\mathrm{Pic}}_{X/k},0} = \{[L] \in \underline{\mathrm{Pic}}_{X/k}(S); e^*(L) = \mathcal{O}_X\}.$$

Show that we have a canonical isomorphism $T_{\underline{\mathrm{Pic}}_{X/k},0} \simeq H^1(X, \mathcal{O}_X)$.

(ii) For any fpqc-morphism $f : T' \to T$ of k-schemes, we have an exact sequence of abelian groups

$$0 \to \underline{\mathrm{Pic}}_{X/k}(T) \xrightarrow{f^*} \underline{\mathrm{Pic}}_{X/k}(T') \xrightarrow{p_1^* - p_2^*} \underline{\mathrm{Pic}}_{X/k}(T' \times_T T').$$

(Hint: use rigidified line bundles to represent elements in $\underline{\mathrm{Pic}}_{X/k}$, and the key point is the automorphism group of a rigidified line bundle is trivial. So by fpqc-descent, ...)

2. Let X be an abelian variety, and L be a line bundle on X. Show that $L \otimes (-1_X)^*(L^{-1})$ is in $\underline{\mathrm{Pic}}^0_{X/k}(k)$.

3. Let $f : X \to Y$ be a proper morphism of locally noetherian schemes, and $\mathcal{F}$ be a coherent sheaf on X flat over Y, and $y \in Y$. We denote by $X_y = f^{-1}(y)$ the fiber of f over y, and by $\mathcal{F}_y$ the pullback of $\mathcal{F}$ to X_y. Show that $H^q(X_y, \mathcal{F}_y) = 0$ for any $q \geq n$ if and only if $(R^q f_* \mathcal{F})_y = 0$ for $q \geq n$.

4. Let X be an abelian variety of dimension n over k. Prove that $H^n(X, \mathcal{O}_X)$ is one dimensional. (Hint: use Serre daulity.)

5. (Koszul complex) **This exercise will be used in the course of next week.**

Let A be a noetherian local commutative ring with maximal ideal $\mathfrak{m}_A$. We say that a sequence of elements $x_1, \cdots, x_n \in \mathfrak{m}_A$ is regular, if for any $1 \leq i \leq n$ the image of x_i in $A/(x_1, \cdots, x_{i-1})$ (with $x_0 = 0$) is non-zero divisor. Let $x_1, \cdots, x_n$ be a regular sequence of A. The Koszul complex associated with $(x_1, \cdots, x_n)$, denoted by $\mathrm{Kos}^\bullet(x_1, \cdots, x_n)$, is defined as

$$0 \to \wedge^n(A^n) \xrightarrow{d_n} \wedge^{n-1}(A^n) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} A^n \xrightarrow{d_1} A \to 0.$$

Here $\wedge^r(A^n)$ is placed at degree $-r$, and $d_r : \wedge^r(A^n) \to \wedge^{r-1}(A^n)$ is given by

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r} \mapsto \sum_{j=1}^{r} (-1)^{j-1} x_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots e_{i_r},$$

where $(e_i)_{1 \leq i \leq n}$ is the standard basis of A^n, and $\hat{e}_{i_j}$ means it's omitted in the wedge.

(i) Show that $H^q(\mathrm{Kos}^\bullet(x_1, \cdots, x)) = 0$ if $q \neq 0$ and $H^0(\mathrm{Kos}^\bullet(x_1, \cdots, x_n)) = A/(x_1, \cdots, x_n)$.

(ii) For a complex $M^\bullet$ of A-modules, we denote by M^r the A-module at degree r. Let $L^\bullet = \mathrm{Kos}^\bullet(x_1, \cdots, x_n)^\vee$ be the dual complex of $K^\bullet = \mathrm{Kos}^\bullet(x_1, \cdots, x_n)$, *i.e.*, $L^\bullet$ is the complex concentrated in degrees $[0, n]$ such that $L^r = \mathrm{Hom}_A(K^{-r}, A)$ for any $r \in \mathbf{Z}$ and $d^r : L^r \to L^{r+1}$ is the morphism induced by $d_{r+1} : K^{-r-1} = \wedge^{r+1}(A^n) \to K^{-r} = \wedge^r(A^n)$. Show that one has a duality of the Koszul complex: $L^\bullet = K^\bullet[-n]$.