Tsinghua 2010 **Abelian Varieties** Problem Set 7

1. Let K be an infinite field, V be a K-vector space, $f\colon V \to K$ be a function such that for all $v, w \in V$, the function $K \to K$ given by $x \mapsto f(xv + w)$ is a polynomial with coefficients in K. Show that f is a polynomial function over K.

Let k be a field. For any abelian variety A over k and any line bundle $\mathcal{L}$ on A, we denote by $\phi_{\mathcal{L}}\colon A \to A^\vee$ the homomorphism given by $a \mapsto T_a^*\mathcal{L} \otimes \mathcal{L}^{\otimes -1}$, where T_a is the translation by a.

2. Let $h\colon B \to A$ be a homomorphism of abelian varieties. Show that $\phi_{h^*\mathcal{L}} = h^\vee \circ \phi_{\mathcal{L}} \circ h$.

A *polarized abelian variety* is defined to be a couple (A, ϕ) consisting of an abelian variety A and a polarization ϕ of A (that is, an isogeny $A \to A^\vee$ such that $\phi_{\bar{k}} = \phi_{\mathcal{L}}$ for some ample line bundle $\mathcal{L}$ on $A_{\bar{k}}$). A morphism of polarized abelian varieties $(B, \psi) \to (A, \phi)$ is a homomorphism $h\colon B \to A$ such that $\psi = h^\vee \circ \phi \circ h$.

3. Let (A, ϕ) be a polarized abelian variety. Show that the group of automorphisms of (A, ϕ) is finite. (*Hint:* rewrite the condition $\phi = h^\vee \circ \phi \circ h$ using Rosati involution and apply the positivity of the latter.)