Odds and ends on equivariant cohomology and traces

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Introduction

Let k be an algebraically closed field of characteristic $p \geq 0$,
X be a separated scheme of finite type over k,
G be a finite group acting on X.
For any prime number $\ell \neq p$, $H^i(X, \mathbb{Q}_\ell)$ is a finite-dimensional ℓ-adic representation of G. For $g \in G$,

$$t_\ell(g) := \sum_i (-1)^i \operatorname{Tr}(g, H^i(X, \mathbb{Q}_\ell)) \in \mathbb{Z}_\ell.$$

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Problem

Is $t_\ell(g)$ in $\mathbb{Z}$ and independent of ℓ?

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Describe the virtual representation $\chi(X, G, \mathbb{Q}_\ell) := \sum_i (-1)^i [H^i(X, \mathbb{Q}_\ell)]$ of G under suitable assumptions on the action of G.

Plan of the talk

1. Generalization of Laumon's theorem on Euler characteristics
2. Tameness at infinity
3. Mod ℓ equivariant cohomology algebra

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Compactly supported cohomology

$H_c^i(X, \mathbb{Q}_\ell)$ is also a finite-dimensional ℓ-adic representation of G.
For $g \in G$,

$$t_{c,X,\ell}(g) := \sum_i (-1)^i \operatorname{Tr}(g, H_c^i(X, \mathbb{Q}_\ell)) \in \mathbb{Z}_\ell.$$

Additivity: If $Z \subset X$ is a G-stable closed subscheme, then

$$t_{c,X,\ell}(g) = t_{c,Z,\ell}(g) + t_{c,X-Z,\ell}(g).$$

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Theorem (Deligne-Lusztig 1976)

$t_{c,\ell}(g)$ is in $\mathbb{Z}$ and independent of ℓ.

Theorem

(1) $$t_\ell(g) = t_{c,\ell}(g).$$

Corollary

$t_\ell(g)$ is in $\mathbb{Z}$ and independent of ℓ.

If $g = 1$, (**??**) becomes

$$\chi(X, \mathbb{Q}_\ell) = \chi_c(X, \mathbb{Q}_\ell),$$

which follows from Laumon's theorem.

Laumon's theorem

Let k be an arbitrary field of characteristic p.
For X separated of finite type over k,
let $D_c^b(X, \mathbb{Q}_\ell)$ be the category of (bounded) ℓ-adic complexes, $K(X, \mathbb{Q}_\ell)$ be the corresponding Grothendieck ring, $K^\sim(X, \mathbb{Q}_\ell)$ be the quotient of $K(X, \mathbb{Q}_\ell)$ by the ideal generated by $[\mathbb{Q}_\ell(1)] - [\mathbb{Q}_\ell]$.
For any morphism $f: X \to Y$, the exact functors

$$Rf_*, Rf_! \colon D_c^b(X, \mathbb{Q}_\ell) \to D_c^b(Y, \mathbb{Q}_\ell)$$

induce group homomorphisms

$$f_*, f_! \colon K(X, \mathbb{Q}_\ell) \to K(Y, \mathbb{Q}_\ell),$$
$$f_*^\sim, f_!^\sim \colon K^\sim(X, \mathbb{Q}_\ell) \to K^\sim(Y, \mathbb{Q}_\ell).$$

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Theorem (Laumon 1981)

$$f_*^\sim = f_!^\sim.$$

Equivariant complexes

For X separated of finite type over k and G finite acting on X, let $D_c^b(X, G, \mathbb{Q}_\ell)$ be the category of (bounded) G-equivariant ℓ-adic complexes, $K(X, G, \mathbb{Q}_\ell)$ be the corresponding Grothendieck ring, $K^\sim(X, G, \mathbb{Q}_\ell)$ be the quotient of $K(X, G, \mathbb{Q}_\ell)$ by the ideal generated by $[\mathbb{Q}_\ell(1)] - [\mathbb{Q}_\ell]$.
Let $(f, u)\colon (X, G) \to (Y, H)$, where $u\colon G \to H$ is a homomorphism and $f\colon X \to Y$ is a u-equivariant morphism. The exact functors

$$R(f, u)_*, R(f, u)_!\colon D_c^b(X, G, \mathbb{Q}_\ell) \to D_c^b(Y, H, \mathbb{Q}_\ell)$$

induce group homomorphisms

$$(f, u)_*, (f, u)_!\colon K(X, G, \mathbb{Q}_\ell) \to K(Y, H, \mathbb{Q}_\ell),$$
$$(f, u)_*^\sim, (f, u)_!^\sim\colon K^\sim(X, G, \mathbb{Q}_\ell) \to K^\sim(Y, H, \mathbb{Q}_\ell).$$

Theorem

$$(f, u)_*^{\sim} = (f, u)_!^{\sim}.$$

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One key step of the proof is the following.

Proposition

Let S be the spectrum of a henselian discrete valuation ring, with closed point $s = \mathrm{Spec}(k)$ and generic point η. Then for any $L \in D_c^b(X \times_s \eta, G, \mathbb{Q}_\ell)$, the class in $K^{\sim}(X, G, \mathbb{Q}_\ell)$ of

$$R\Gamma(I, L) \in D_c^b(X, G, \mathbb{Q}_\ell)$$

is zero, where I is the inertia subgroup of the Galois group of η.

Complexes on Deligne-Mumford stacks

Let S be a regular (Noetherian) scheme of dimension ≤ 1,
ℓ be a prime invertible on S.
One can define, for every Deligne-Mumford stack $\mathcal{X}$ of finite type over S, a category $D_c^b(\mathcal{X}, \mathbb{Q}_\ell)$ of ℓ-adic complexes on $\mathcal{X}$, and, for every morphism $f\colon \mathcal{X} \to \mathcal{Y}$, exact functors

$$Rf_*, Rf_!\colon D_c^b(\mathcal{X}, \mathbb{Q}_\ell) \to D_c^b(\mathcal{Y}, \mathbb{Q}_\ell),$$
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Under an additional condition of finiteness of cohomological dimension, Laszlo-Olsson 2008 defined an ℓ-adic formalism for unbounded complexes on Artin stacks.

For a Deligne-Mumford stack $\mathcal{X}$ of finite type over S,
let $K(\mathcal{X}, \mathbb{Q}_\ell)$ be the Grothendieck ring of $D_c^b(\mathcal{X}, \mathbb{Q}_\ell)$,
$K^\sim(\mathcal{X}, \mathbb{Q}_\ell)$ be the quotient by the ideal generated by $[\mathbb{Q}_\ell(1)] - [\mathbb{Q}_\ell]$.
For $f: \mathcal{X} \to \mathcal{Y}$, Rf_* and $Rf_!$ induce group homomorphisms

$$f_*^\sim, f_!^\sim : K^\sim(\mathcal{X}, \mathbb{Q}_\ell) \to K^\sim(\mathcal{Y}, \mathbb{Q}_\ell),$$

f^* and $Rf^!$ induce group homomorphisms

$$f^{*\sim}, f^{!\sim} : K^\sim(\mathcal{Y}, \mathbb{Q}_\ell) \to K^\sim(\mathcal{X}, \mathbb{Q}_\ell).$$

$f^{*\sim}$ is a ring homomorphism.

For a Deligne-Mumford stack $\mathcal{X}$ of finite type over S,
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$$f_*^\sim = f_!^\sim.$$

Corollary

$$f^{*\sim} = f^{!\sim}.$$

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Free actions

From now on, k is an algebraically closed field of characteristic exponent $p \geq 1$. For X separated of finite type over k and G finite acting on X, $t(g) := t_\ell(g) = t_{c,\ell}(g) \in \mathbb{Z}$ for $g \in G$.

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Theorem

If G acts freely on X, then $R\Gamma(X, \mathbb{Z}_\ell)$ and $R\Gamma_c(X, \mathbb{Z}_\ell)$ are perfect complexes of $\mathbb{Z}_\ell[G]$-modules and $t(g) = 0$ for every $g \in G$ whose order is not a power of p.

A complex of $\mathbb{Z}_\ell[G]$-modules is said to be perfect if it is quasi-isomorphic to a bounded complex of finite projective $\mathbb{Z}_\ell[G]$-modules.
If P is a finite projective $\mathbb{Z}_\ell[G]$-module, the theory of modular characters implies that the character $P \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ vanishes on ℓ-singular elements of G, namely, elements of order divisible by ℓ.

Corollary (Deligne-Lusztig (for χ_c))

If G acts freely on X, and the order of G is prime to p, then

$$\chi(X, G, \mathbb{Q}_\ell) = \chi(X/G)\mathrm{Reg}_{\mathbb{Q}_\ell}(G),$$

where $\mathrm{Reg}_{\mathbb{Q}_\ell}(G)$ *is the regular representation of G.*

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Corollary (Serre)

If G is an ℓ-group acting on X, then

$$\chi(X^G) \equiv \chi(X) \mod \ell.$$

Example

If G is an ℓ-group acting on $X = \mathbb{A}^n$, then $\chi(X^G) \equiv 1 \mod \ell$.
In particular, $X^G \neq \emptyset$.

Let Y be a connected normal separated scheme of finite type over k.
We say a finite Galois etale cover $X \to Y$ of group G is tamely ramified at infinity, if there exists a normal compactification $\bar{Y}$ of Y, such that, at every point x of the normalization $\bar{X}$ of $\bar{Y}$ in X,

$$\begin{array}{ccc} X & \hookrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & \bar{Y} \end{array}$$

the inertia subgroup of G has order prime to p.

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Theorem (Deligne-Illusie 1981)

Under the above assumptions,

$$\chi(X, G, \mathbb{Q}_\ell) = \chi(X/G)\mathrm{Reg}_{\mathbb{Q}_\ell}(G).$$

Vidal's group

Let Y be a connected normal separated scheme of finite type over k, $\bar{\zeta}$ be a geometric point of Y. Define the wild part E_Y of $\pi_1(Y, \bar{\zeta})$:

$$E_Y = \bigcap_{\bar{Y}} E_{Y,\bar{Y}},$$

where $\bar{Y}$ runs over compactifications of Y, $E_{Y,\bar{Y}}$ is the closure of $\cup_{\bar{y}\in\bar{Y}} E'_{\bar{y}}$, where $E'_{\bar{y}}$ is the union of the images of the p-Sylows of $\pi_1(\bar{Y}_{(\bar{y})} \times_{\bar{Y}} Y)$ in $\pi_1(Y, \bar{\zeta})$.

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For $a \in K_{\mathrm{lisse}}(Y, \mathbb{F}_\ell)$, the Brauer trace map

$$\mathrm{Tr}_a^{\mathrm{Br}} \colon \pi_1(Y, \bar{\zeta})_{\ell\text{-reg}} \to \mathbb{Z}_\ell$$

is defined on ℓ-regular elements, namely, elements of profinite order prime to ℓ.

Let Z be a scheme separated of finite type over k, $K(Z, \mathbb{F}_\ell)$ be the Grothendieck group of constructible sheaves of $\mathbb{F}_\ell$-modules on Z. Vidal 2004 defines

$$K(Z, \mathbb{F}_\ell)^0_t \subset K(Z, \mathbb{F}_\ell)$$

as the subgroup generated by classes of the form $[i_! a]$, where $i\colon Y \to Z$ is a quasi-finite morphism, Y is a connected normal separated scheme, $a \in K_{\mathrm{lisse}}(Y, \mathbb{F}_\ell)$, $\mathrm{Tr}^{\mathrm{Br}}_a(s) = 0$ for all $s \in E_Y$.

We extend this definition verbatim to algebraic spaces Z separated of finite type over k.

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We extend this definition verbatim to algebraic spaces Z separated of finite type over k.

Theorem (Gabber-Vidal 2005)

Let Y be a connected normal separated scheme of finite type over k, $a \in K_{\mathrm{lisse}}(Y, \mathbb{F}_\ell)$. Then a is in $K(Y, \mathbb{F}_\ell)^0_t$ if and only if there exists a compactification $\bar{Y}$ of Y such that $\mathrm{Tr}^{\mathrm{Br}}_a(s) = 0$ for all $s \in E_{Y,\bar{Y}}$.

Virtual tameness

Let Z be an algebraic space separated of finite type over k.
The rank function is a homomorphism

$$\mathrm{rank}\colon K(Z, \mathbb{F}_\ell) \to C(Z, \mathbb{Z}),$$

where $C(Z, \mathbb{Z})$ is the group of constructible functions on Z.
It has a section $c \mapsto \langle c \rangle$, sending the characteristic function of any locally closed subspace Z' of Z to $i_! \mathbb{F}_{\ell, Z'}$, where $i\colon Z' \to Z$ is the immersion.
We say $a \in K(Z, \mathbb{F}_\ell)$ is virtually tame if $a - \langle \mathrm{rank}(a) \rangle \in K(Z, \mathbb{F}_\ell)^0_t$.

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Proposition

Let Y be a connected normal separated scheme of finite type over k, $f\colon X \to Y$ be a finite Galois etale cover. Then f is tamely ramified at infinity if and only if $[f_* \mathbb{F}_\ell] \in K(Y, \mathbb{F}_\ell)$ is virtually tame.

Let X be a scheme separated of finite type over k,
G be a finite group acting on X.
Then X/G is a separated algebraic space of finite type over k.
We say the action is virtually tame
if $[f_*\mathbb{F}_\ell] \in K(X/G, \mathbb{F}_\ell)$ is virtually tame, where $f\colon X \to X/G$.

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For $H < G$, let $X_H = X^H - \cup_{H<H'<G} X^{H'}$ be the locus of inertia H.
Let $\mathcal{S}$ be the set of conjugacy classes of subgroups of G.
For $S \in \mathcal{S}$, G acts on $X_S := \coprod_{H\in S} X_H$.

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Theorem

Assume that the action of G is virtually tame. Then

$$\chi(X, G, \mathbb{Q}_\ell) = \sum_{S\in\mathcal{S}} \chi(X_S/G) I_S,$$

where $I_S = \mathbb{Q}_\ell[G/H]$ for $H \in S$.

Verdier 1976 proved an analogue for certain locally compact topological spaces (for example X^{an}, if $k = \mathbb{C}$).

Corollary

If $G = \langle g \rangle$ acts virtually tamely on X, then

$$t(g) = \chi(X^g).$$

The case X affine smooth over $\mathbb{C}$ was known to Petrie-Randall 1986.

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Equivariant cohomology algebra

Recall that k is an algebraically closed field.
Let X be a separated scheme of finite type over k,
G be a linear algebraic group over k. Then $[X/G]$ is an Artin stack.
$BG := [\mathrm{Spec}(k)/G]$.
For $L \in D_c^b([X/G], \mathbb{F}_\ell)$, $H^i([X/G], L)$ is a finite-dimensional $\mathbb{F}_\ell$-vector space.
If G is a finite group, $H^i([X/G], L) = H^i(G, R\Gamma(X, L))$.

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If G is a finite group, $H^i([X/G], L) = H^i(G, R\Gamma(X, L))$.

Example

For $A \simeq (\mathbb{Z}/\ell)^r$ (such a group is called an elementary abelian ℓ-group),

$$H^*(BA, \mathbb{F}_\ell) = \begin{cases} \mathbb{F}_\ell[x_1, \dots, x_r] & \ell = 2, \\ \wedge(x_1, \dots, x_r) \otimes \mathbb{F}_\ell[y_1, \dots, y_r] & \ell > 2, \end{cases}$$

where $x_1, \dots, x_r$ form a basis of $H^1 = \mathrm{Hom}(A, \mathbb{F}_\ell)$, $y_1, \dots, y_r \in H^2$.

Finiteness

Theorem

$H^*([X/G], \mathbb{F}_\ell)$ is a finitely generated $\mathbb{F}_\ell$-algebra and $H^*([X/G], L)$ is a finite $H^*([X/G], \mathbb{F}_\ell)$-module.

Topological setting (G compact Lie group, X certain topological space, $L = \mathbb{F}_\ell$) known to Quillen 1971.

The structure theorem

Let $\mathcal{A}$ be the category of pairs (A, c), where A is an elementary abelian ℓ-subgroup of G, $c \in \pi_0(X^A)$. A morphism $(A, c) \to (A', c')$ in $\mathcal{A}$ is a $g \in G(k)$ such that $gAg^{-1} \subset A'$, $gc \supset c'$. $BA \times c \to [X/G]$ induces

$$H^*([X/G], \mathbb{F}_\ell) \to H^*(BA \times c, \mathbb{F}_\ell) \to H^*(BA, \mathbb{F}_\ell).$$

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Theorem

The homomorphism

$$H^*([X/G], \mathbb{F}_\ell) \to \varprojlim_{(A,c)\in\mathcal{A}} H^*(BA, \mathbb{F}_\ell)$$

is a uniform F-isomorphism.

A homomorphism of $\mathbb{F}_\ell$-algebras is called a uniform F-isomorphism if $F^N = 0$ on the kernel and cokernel for N large enough. Here $F\colon a \mapsto a^\ell$.

Topological setting known to Quillen 1971.

For $A \subset A'$, $BA \to BA'$, $X^A \supset X^{A'}$.
Let $\mathcal{B}$ be the category of pairs (A, Z), where $A \subset Z \subset G$ are elementary abelian ℓ-subgroups. A morphism $(A, Z) \to (A', Z')$ in $\mathcal{B}$ is a $g \in G(k)$ such that $gAg^{-1} \subset A'$, $gZg^{-1} \supset Z'$. 2-commutative diagram:

$$\begin{array}{ccc} BA \times X^Z & \longrightarrow & BA' \times X^{Z'} \\ & \searrow & \downarrow \\ & & [X/G] \end{array}$$

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Theorem

Let $L \in D_c^b([X/G], \mathbb{F}_\ell)$, endowed with a ring structure $L \otimes L \to L$. Then the homomorphism

$$H^*([X/G], L) \to \varprojlim_{(A,Z) \in \mathcal{B}} H^*(BA \times X^Z, L)$$

is a uniform F-isomorphism.

The end

Thank you.